Higher dimensional uniformisation and W-geometry

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Abstract

We formulate the uniformisation problem underlying the geometry of W_n-gravity using the differential equation approach to W-algebras. We construct W_n-space (analogous to superspace in supersymmetry) as an $(n-1)$ dimensional complex manifold using isomonodromic deformations of linear differential equations. The W_n-manifold is obtained by the quotient of a Fuchsian subgroup of $\mathrm{PSL}(n,\mathbb{R})$ which acts properly discontinuously on a simply connected domain in $\mathbb{CP}^{n-1}$. The requirement that a deformation be isomonodromic furnishes relations which enable one to convert non-linear W-diffeomorphisms to (linear) diffeomorphisms on the W_n-manifold. We discuss how the Teichmüller spaces introduced by Hitchin can then be interpreted as the space of complex structures or the space of projective structures with real holonomy on the W_n-manifold. The projective structures are characterised by Halphen invariants which are appropriate generalisations of the Schwarzian. This construction will work for all "generic" W-algebras.

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1 Introduction

The search for higher spin extensions of the Virasoro algebra led to the introduction of W_3-algebra[1]. Unlike the Virasoro algebra, this algebra turned out to be non-linear. W-algebras associated with various simple Lie groups have been constructed. One of the methods has been obtained from the work of Drinfeld and Sokolov[2] who associated differential equations with Lie Algebras and showed that the "coefficients" of the linear differential equations satisfy classical W-algebras when equipped with the Gelfand-Dikii Poisson bracket. For a complete list of references, see the reviews by Bouwknegt and Schoutens, and Feher *et al.*[3].

It is well known that the generators of the Virasoro algebra occur as the generators of $Diff(S^1)$ as well as the residual diffeomorphisms in the conformal gauge in two dimensional gravity (where one gets two copies of the Virasoro algebra). It is natural to ask what sort of structure underlies the plethora of classical W-algebras. It is not clear what replaces $Diff(S^1)^1$. In analogy with the Virasoro case, the generators of W-algebras should generate residual W-diffeomorphisms of W-gravity. Extending this further, one can construct W-string theories. W-gravity defined this way is confined to a particular gauge and the meaning of covariance is not very clear. Recently, there has been some progress in this regard using a relationship to Higgs bundles[4, 5]. W-gravity can also be formulated as the theory of symmetric tensors (with the metric tensor as the first in the series). This approach has been pursued by Hull[6].

In the Polyakov path integral approach to string theory, one integrates over all metrics modulo those which are equivalent under diffeomorphisms and Weyl transformations. On a Riemann surface Σ with genus $g > 1$, this integral reduces to a finite dimensional integral over the moduli space of Riemann surfaces which is a $(3g-3)$ dimensional complex space. Related to this is the Teichmüller space of Riemann surfaces which when quotiented by "large" diffeomorphisms (i.e., diffeomorphisms not connected to the identity) gives the moduli space. One expects that such a space should exist even for the case of W-strings. In a recent paper[5], it was shown that the Teichmüller space for W-gravity is the same as one of the components of the moduli space of certain stable Higgs bundles studied by Hitchin[7]. Hitchin calls this space the *Teichmüller component.* These spaces are a particular component of the space $\text{Hom}(\pi_1(\Sigma); G)/G$ where G is a simple Lie group. For $G = \text{PSL}(n, \mathbb{R})$, we shall label the Teichmüller component by $\mathcal{T}_n$ with $\mathcal{T}_2$ being the usual Teichmüller space of Riemann surfaces.

The Teichmüller space of Riemann surfaces, $\mathcal{T}_2$ is associated with the uniformisation of Riemann surfaces. Since $\mathcal{T}_2$ is the Teichmüller component of $\text{Hom}(\pi_1(\Sigma); G)/G$ with $G = \text{PSL}(2, \mathbb{R})$, every element of this space furnishes a Fuchsian subgroup of $\text{PSL}(2, \mathbb{R})$ which is isomorphic to the fundamental group of Σ. This Fuchsian group acts properly discontinuously on the half-plane with Poincaré metric and the Riemann surface Σ can be represented as the quotient of the half-plane by the Fuchsian group. This is the uniformisation theorem for surfaces of genus $g > 1$.

It is natural to ask what kind of uniformisation is associated with the other Teichmüller spaces $\mathcal{T}_n$? This paper attempts to answer this question. From the work of ref. [5], it is clear that this question is also intimately related to the geometry of W-gravity. We choose to address this question using the differential equation approach to uniformisation. For the case of the usual uniformisation of Riemann surfaces, this was studied by Poincaré and more recently by Hejhal[8]. One can associate a monodromy group Γ to an n-th order Fuchsian differential equation[2]. Generically, Γ is a subgroup of $\text{PSL}(n, \mathbb{C})$. By choosing a situation such that the monodromy group Γ is an element of $\mathcal{T}_n$, we end up having a situation where it is a subgroup of $\text{PSL}(n, \mathbb{R})$.

The differential equation furnishes (locally) a map from the Riemann surface to $\mathbb{CP}^{n-1}$. Lifting the differential equation to the universal cover of the Riemann surface (which we shall model by the half-plane with Poincaré metric), we obtain a map from the half-plane to $\mathbb{CP}^{n-1}$. Let the image of the half-plane

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[1] The results of this paper suggest that S^1 should also be extended to a higher dimensional manifold using the KdV times as extra dimensions in order to understand the action of W-diffeomorphisms.

[2] On analytically continuing solutions of the differential equation along non-trivial loops on the surface, the solutions go into linear combinations of each other. So one can associate a monodromy matrix for every loop. The analyticity of solutions implies that this matrix depends only on the homotopy class of the loop. The set of the monodromy matrices form the monodromy group Γ.

be represented by $\Omega \subset \mathbb{CP}^{n-1}$. The monodromy group acts properly discontinuously on Ω. So we create a W-surface by the quotient Ω/Γ. There exist a set of deformations of the differential equation which preserve Γ – these are the isomonodromic deformations and they are parametrised by the times of the generalised KdV hierarchy associated with any linear differential operator. Labelling the first $(n-1)$ times $(t_1 = z, t_2, \ldots, t_{n-1})$, we obtain a $\Omega(\{t_i\})$ for every time slice with the same group Γ acting properly discontinuously on them. We form the $(n-1)$ dimensional complex manifold $\widetilde{\Omega}$ by the union of all the $\Omega(\{t_i\})$. We form the W-manifold by the quotient $\widetilde{\Omega}/\Gamma$. This manifold which we shall call $W_n(\Sigma)$ provides us with W-space with the KdV times as the w-coordinates. The uniformisation we have just described is the main result of this paper. A special set of diffeomorphisms on this manifold project down to the W-surface as W-diffeomorphisms.

The possibility of using the KdV times to linearise diffeomorphisms is not new[9, 10, 11]. However, what is new is the use of the fact that they provide isomonodromic deformations. This makes the use of KdV times more natural especially in the context of Fuchsian uniformisation. It is natural to restrict diffeomorphisms on the W-manifold to those that preserve the form of the equations (2.18) which imply isomondromic deformations. This method has been used recently by Gomis *et al.*. They have obtained finite W-transformations and have shown that this set of diffeomorphisms close under composition i.e., form a semi-group[11]. This is indeed a remarkable result. This paper[11] does have some overlap with the present work. However, since our approaches are distinct, the results complement each other.

There has been a large amount of work done in the context of W-geometry. We shall briefly mention some of them. One of the earliest is the work of Sotkov and Stanishkov[12] who interpreted W-geometry as affine geometry. Some interesting observations have also been made in the Cargese lectures of Itzykson[13]. Gervais and Matsuo have studied W-geometry in the context of Toda theories and discuss embeddings into $\mathbb{CP}^n$. They associate singularities of the embeddings with global indices[10]. But their work does not seem to be related to the Fuchsian uniformisation presented in this paper.[3] Alvarez-Gaumé discusses

the problem of uniformisation in his ICTP lectures[14] and more recently, Aldrovandi and Falqui have also attempted to understand uniformisation using period maps[15].

We now summarise the results of this paper. The key result of the paper is the formulation of a Fuchsian uniformisation related to the Teichmüller spaces of Hitchin. Other results are: A direct relationship to W-gravity is presented by relating the Beltrami differential's of the uniformisation to the Beltrami differentials of W-gravity (in sec. 4) and thus recovering W-diffeomorphisms from a subgroup of diffeomorphisms on the higher dimensional manifold. This proves that the manifolds obtained from uniformisation can be considered as w-space. We also discuss how higher dimensional generalisations of the Schwarzian, which we construct play the role of projective connections on the higher dimensional manifold.

The paper is organised as follows. In section 2, we discuss the basics of linear differential equations and introduce Halphen invariants which are generalisations of the Schwarzian. We then discuss the conditions under which a deformation of a linear differential equation is isomonodromic. This section is mainly a collection of well known results and are included for completeness and to fix the notation. In section 3, we formulate the uniformisation problem and construct W-space. In section 4, we discuss how deformations of complex structure in W-space project to give the standard Beltrami equations on the W-surface. We also show how deformations of complex structure can be parametrised by higher symmetric differentials. In section 5, we briefly discuss the projective structures on W-manifolds and present higher dimensional generalisations of the Schwarzian. In section 6, we end with some conclusions and suggestions.

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[3]The difference can be seen by considering the well understood case of Fuchsian uniformisation of compact Riemann surfaces. Here the uniformisation map is not an embedding but a multi-valued (polymorphic) map with no singularities. Another difference, is that the natural metric related to uniformisation is not the Fubini-Study metric considered in [10] but the hyperbolic Poincaré metric. Thus the maps to $\mathbb{CP}^n$ discussed in this paper are not the same as those discussed by Gervais and Matsuo. It would however be of interest to interpret their results in the present context.

2 Linear differential equations

The relationship of linear differential equations to W-algebras is quite well known. We refer the reader to the work of Drinfeld and Sokolov[2] and Di Francesco *et al.*[16] for details. In this section, we shall introduce Halphen invariants which are generalisations of the Schwarzian and then discuss isomonodromic deformations of linear differential equations.

2.1 Basics

Consider the homogeneous linear differential operator L given by

$$\hat{L} \equiv D^n + \binom{n}{1}\hat{u}_1(z) \; D^{n-1} + \cdots + \hat{u}_n(z) \quad , \tag{2.1}$$

where $D = \frac{d}{dz}$. The differential equation

$$\hat{L} \; \hat{f} = 0 \quad , \tag{2.2}$$

has locally n linearly independent solutions which we shall label by $\{f_i\}$. We shall assume that it is of the Fuchsian type, i.e., it has only a finite number of regular singular points and no irregular singular points. (This implies that the $\hat{u}_i$ have at most poles of order i at the singular points and are analytic elsewhere.) In this paper, we shall be restrict ourselves to the case of no singularities. However, we shall be solving these differential equations on Riemann surfaces of genus larger than one and thus the non-triviality comes from the non-trivialness of its fundamental group. Singularities correspond to punctures (on the Riemann surface) and we shall not discuss them here. By means of the following redefinition $f = exp(\int \hat{u}_1) \; \hat{f}$, the differential equation can always be converted to one of the form

$$L \, f = 0 \quad , \tag{2.3}$$

where

$$L \equiv D^n + \binom{n}{2}u_2(z) \; D^{n-2} + \cdots + u_n(z) \quad . \tag{2.4}$$

The vanishing of the coefficient of D^{n-1} implies that the Wronskian is a constant (chosen to be equal to 1 with no loss of generality). For this to be consistent globally on the Riemann surface, the Wronskian has to transform like a scalar.

A simple calculation (see appendix A) shows that this implies that f is a $\frac{(1-n)}{2}$ – differential. For example, for $n = 2$, this differential equation has an important role in the uniformisation of Riemann surfaces. We obtain the well known result that $(-\frac{1}{2})$ – differentials play a crucial role in the context of Riemann surfaces[17]. The transformation properties of the u_i under holomorphic change of the independent variable z is somewhat complicated except for u_2 which transforms like the Schwarzian. However, there exists an invertible (non-linear in u_2) change of variables $u_i \rightarrow w_i$ where w_i for $i > 2$ transform as tensors of weight i. This change of variables is linear when $u_2 = 0$ and uniquely fixed by the tensor weight of w_i and the linear dependence on u_i[19, 16]. When $u_2 = 0$, one can explicitly write out the expression for w_i in terms of u_i and its derivatives.

$$w_m = \frac{1}{2}\sum_{s=0}^{m-3}(-)^s \frac{(m-2)!m!(2m-s-2)!}{(m-s-1)!(m-s)!(2m-3)!s!}u_{m-s}^{(s)} \quad , \tag{2.5}$$

for $m = 3, \ldots, n$. It is interesting to note that the above expressions are the same for all n. This explains the unusual choice normalisations in (2.3). The dependence on u_2 which is non-linear can be restored using an algorithm described in [16].

One can always choose a coordinate system with $u_2 = 0$. The differential equation (2.3) in this coordinate system is said to be of the *Laguerre-Forsyth* form. This coordinate system is obtained from the solutions of the following auxiliary second order differential equation

$$(D^2 + \frac{3}{n+1}u_2)\theta = 0 \quad . \tag{2.6}$$

Let θ_1, θ_2 be two linear independent solutions of this equation. The coordinate system where $u_2 = 0$ corresponds to the coordinate $w = \theta_1/\theta_2$. The residual transformations which preserve the Laguerre-Forsyth form are the Möbius transformations.

When all $w_i = 0$ for $i > 2$, all solutions of (2.3) can be generated from the two solutions $\{\theta_1, \theta_2\}$ of the auxiliary differential equation (2.6). The solutions are given by θ_1^{n-1}, $\theta_1^{n-2}\theta_2$, $\ldots$, θ_2^{n-1}. (This is the generalisation of the 1, z, $\ldots$, z^{n-1} when all $u_i = 0$.) Further, all the u_i for $i > 2$ are completely

determined in terms of u_2 and its derivatives. For example, $u_3 = \frac{3}{2}u_2'$ and $u_4 = \frac{9}{5}u_2'' + \frac{3(5n+7)}{5(n+1)}u_2^2$[18].

For the case of $n = 3$, it is easy to see that the solutions of the differential equation satisfy the quadratic relation

$$f_1 f_3 - (f_2)^2 = 0 \quad . \tag{2.7}$$

The subgroup of PSL$(3, \mathbb{C})$ that preserves the relation is PSL$(2, \mathbb{C})$.

2.2 Schwarzian and it generalisations

The $\{f_i\}$ can be thought of (locally) as homogeneous coordinates on $\mathbb{CP}^{n-1}$. We can define inhomogeneous coordinates s_i for $i = 1, \ldots, (n-1)$ by $s_i = (f_i/f_n)$. The $\{u_i\}$ now play the role of *projective invariants* which characterise the map into $\mathbb{CP}^{n-1}$. Projective invariants are differential functions $I^r(\{s_i\}; z)$ which are invariant under the group of projective transformations. For example, for the case of $n = 2$, the Schwarzian is the invariant. We shall now show how one constructs these invariants. We shall follow the construction described in Forsyth[18] and Wilczynski[19]. In his study of plane and space curves (n=3,4 in our language), Halphen constructed quantities from the inhomogeneous co-ordinates which were invariant under projective transformations. Hence the I^r are called the *Halphen invariants* by Wilczynski. Painlevé and Boulanger also derived the invariants for $n = 3$. These have been derived more than a century ago in the study of classical invariant theory.

The Schwarzian will be denoted by $S(s; z)$. It has the following properties

$$S(\frac{as+b}{cs+d}; z) = S(s; z) \quad ,$$
$$S(s; z) = S(s; t)(\frac{dt}{dz})^2 + S(t; z) \quad , \tag{2.8}$$

where the first equation is the statement of projective invariance of the Schwarzian and the second gives its transformation under holomorphic change of coordinates. Under Möbius transformations, it shows that the Schwarzian behaves as a modular form of degree two.

The projective invariants I^r for $r = 2, \ldots, n$ have the following properties

(Compare with (2.8).)

$$I^r(A\vec{s}; z) = I^r(s; z) \quad ,$$
$$I^2(\vec{s}; z) = I^2(\vec{s}; t)(\frac{dt}{dz})^2 + \frac{n+1}{6}S(t; z) \quad ,$$
$$I^r(\vec{s}; z) = I^r(\vec{s}; t)(\frac{dt}{dz})^r \quad , \quad n > 3 \tag{2.9}$$

where the superscript r indicates the weight of the invariant under transformations of the independent variable and $A \in$ PSL$(n, \mathbb{C})$. It will be shown that solutions of $n-th$ order differential equation (2.3) give rise to solutions of the following coupled set of non-linear differential equations

$$I^r(\vec{s}; z) = w_r(z) \quad . \tag{2.10}$$

The invariance under projective transformations, $\vec{s} \to A\vec{s}$, can be seen from the projective invariance of the right hand side of the above equation. For the case $n = 2$ this gives the well known result that solutions of the non-linear differential equation $S(f; z) = u_2(z)$ can be constructed from the ratio of any two solutions of the second order differential equation (2.3).

The differential invariants are constructed as follows[18]. We shall explicitly illustrate the steps using the $n = 2$ case. Substitute $f_i = s_i f_n$ for $i = 1, \ldots, (n-1)$ in the differential equation (2.3). We obtain $(n-1)$ equations (call them Set A) linear and homogeneous in f_n and its derivatives.

$$\text{SetA}: \quad s'' f_2 + 2s' f_2' = 0 \quad .$$

By differentiating these equations, we obtain $(n-1)$ equations (call them Set B) linear and homogeneous in f_n and its first $(n-1)$ derivatives.

$$\text{SetB}: \quad (s''' - 2s' u_2)f_2 + 3s'' f_2' = 0 \quad .$$

Choose one equation from Set B and all of Set A to obtain n linear equations in f_n and its derivatives. The existence of non-trivial solutions f_n implies that the determinant of the matrix of the coefficients of f_n and its derivatives vanish. We thus obtain $(n-1)$ equations, one for each equation in Set B.

$$\begin{vmatrix} s'' & 2s' \\ (s''' - 2s' u_2) & 3s'' \end{vmatrix} = 0$$

Solving these $(n-1)$ equations for the variables u_i, we obtain expressions for the $u_i \equiv J^i(\vec{s}; z)$ in terms of the inhomogeneous variables and their derivatives. Making the change of variables $u_i \to w_i$, we obtain the expressions $w_r = I^r(\vec{s}; z)$. As is clear, the $n = 2$ case gives the Schwarzian. Explicit expressions for $n = 3$ are given in Forsyth's book[18, see page 195].

There exists a bijective map from the set of solutions of the linear homogeneous differential equation (2.3) and the non-linear differential equation (2.10). The proof of this is a follows. We have just seen, $s_i = f_i/f_n$ for $i = 1, \ldots, (n-1)$ gives us solutions $\vec{s}$ of (2.10) provided f_i solve (2.3). To prove the converse, one can show that

$$J^{-\frac{1}{n}} \equiv f_n = \begin{vmatrix} s_1' & \cdots & s_1^{(n-1)} \\ \vdots & \ddots & \vdots \\ s_{n-1}' & \cdots & s_{n-1}^{(n-1)} \end{vmatrix}^{-\frac{1}{n}} \quad (2.11)$$

and $f_i = s_i f_n$ are solutions of (2.3) provided $\vec{s}$ solve (2.10). The expression for f_n follows from property (A.3) of the Wronskian. Choose $\lambda = f_n$ and using the fact that the Wronskian is a constant (chosen to be 1), we recover the expression for f_n. Hence the solutions of (2.3) can be written in terms of $\vec{s}$ as $f_i = s_i f_n$ for $i = 1, \ldots, (n-1)$ and f_n as given above.

2.3 Isomonodromic Deformations and the generalised KdV hierarchy

Let $\{f_i\}$ for $i = 1, \ldots, n$ be a basis of linearly independent solutions of the differential equation (2.3) with Wronskian equal to 1. Let γ be a closed path (possibly around a regular singular point or a non-trivial cycle with base point z_0). On analytically continuing the solutions f_i around the path γ, we obtain another basis $\{\check{f}_i\}$. Since there are only n independent solutions, the new basis $\{f_i'\}$ is related linearly to the old basis

$$\check{f}_i = [M_\gamma(z_0)]_i{}^j \, f_j \quad,$$

where $M_\gamma(z_0)$ is a constant (independent of z) PGL$(n, \mathbb{C})$ matrix which becomes a PSL$(n, \mathbb{C})$ matrix due to the condition on the Wronskian. The analyticity of the solutions in the neighbourhood of non-singular points implies that $M_\gamma(z_0)$

depends only on the homotopy class of γ and the choice of basis $\{f_i\}$. M_γ are referred to as *monodromy matrices*. The monodromy matrices furnish a map from $\pi_1(\Sigma \backslash E)$ to PSL$(n, \mathbb{C})$, where E denotes the set of regular singular points of (2.3). The set of matrices M_γ form a group which we shall call the *monodromy group*. Further, the freedom in the choice of basis implies that this map is well defined up to an overall conjugation in PSL$(n, \mathbb{C})$. Hence the monodromy of a Fuchsian differential equation furnishes an element of $\text{Hom}(\pi_1; \text{PSL}(n, \mathbb{C}))$ well defined up to overall conjugation in PSL$(n, \mathbb{C})$. We shall refer to the conjugacy class as the *monodromy homomorphism* of the differential equation or simply the monodromy homomorphism.

Let $T_\gamma(z_0)$ be the "generator" of translation along γ. So we have

$$T_\gamma(z_0)\vec{f} = M_\gamma(z_0)\vec{f} \quad. \quad (2.12)$$

Deformations of equation (2.3) such that the monodromy group is preserved are called *isomonodromic deformations*. We shall show that flows of the generalised KdV hierarchy, such that (2.18) is satisfied, generate such deformations. Isomonodromic deformations have been studied in the context of linear differential equations with regular as well as irregular singular points on $\mathbb{CP}^1$ in [20, 21]. We shall however base our discussion on the work of Novikov where he discusses changes of the monodromy matrix under deformations in the context of the second order (Sturm-Liouville) equation with periodic potential[22].

The generalised KdV hierarchy associated with the operator L is defined by the following set of non-linear differential equations in u_i[2].

$$\frac{d}{dt_p}L = [(L^{p/n})_+, L] \quad, \quad (2.13)$$

where $p = 1, 2, \ldots$ and by $(L^{p/n})_+$, we mean the differential operator part of the pseudo-differential operator $(L^{p/n})$. Here the u_i are extended as functions of t_p. For example, in the $n = 3$ case, the u_2 and u_3 satisfy the equations of the Boussinesq hierarchy. The variables t_p represent the "times" of the generalised KdV hierarchy. We will be consider only the first $(n-1)$ flows and one can choose $t_1 = z$.

Taking the derivative of the differential equation $L\vec{f} = 0$ with respect to the

time variable t_p and substituting the flow condition (2.13), we get

$$L \left(\partial_p \vec{f} - (L^{p/n})_+ \vec{f} \right) = 0 \quad . \tag{2.14}$$

Note that we have extended $\vec{f}$ as functions of all the KdV times. This implies that the term in the brackets above also solves the differential equation (2.3). Since $\vec{f}$ give a basis of solutions, we can write

$$(\partial_p \vec{f} - (L^{p/n})_+ \vec{f}) = \Lambda_p \vec{f} \quad , \tag{2.15}$$

where Λ_p is some (z_0 and t_p dependent) matrix. In order to obtain the time dependence of the monodromy matrices $M_\gamma(z_0)$, we will apply the translation operation $T_\gamma(z_0)$ on equation (2.15) to obtain the following equation,

$$\partial_p M_\gamma(z_0) = [\Lambda_p, M_\gamma(z_0)] \quad , \tag{2.16}$$

where Λ_p does not depend on the choice of γ. Since, M is an element of the Lie group $PSL(n, \mathbb{C})$ for all t_p, it follows that Λ_p is an element of the corresponding Lie Algebra. Further, integrating the equation, we obtain

$$M_\gamma(t_p) = g(t_p) \ M_\gamma(0) \ g^{-1}(t_p) \quad , \tag{2.17}$$

where $g(t_p) = P(exp \int_0^{t_p} dt_p \Lambda_p)$ is an element of $PSL(n, \mathbb{C})$. Hence, the conjugacy class of the monodromy homomorphism remains invariant for an arbitrary KdV flow.

When Λ_p is proportional to the identity matrix, we obtain from (2.16) that the monodromy matrices are time independent. This implies that such KdV flows are isomonodromic. Choosing this constant to be zero and substituting in equation (2.15), we get

$$\partial_p \vec{f} = (L^{p/n})_+ \vec{f} \quad . \tag{2.18}$$

Hence, the generalised KdV flows generate isomonodromic deformations when (2.18) is satisfied. This is probably a well known result but we are unaware of a reference providing this result in the context of linear differential equations on a Riemann surface.

3 Higher dimensional uniformisation

The solution of second order Fuchsian differential equations on Riemann surfaces of genus $g > 1$ are related to the uniformisation of Riemann surfaces. This was studied extensively by Poincaré and more recently by Hejhal[8]. This has not been a very successful approach to uniformisation partly due to the problem of the so called accessory parameters. The higher order Fuchsian differential equations should also be related to what we shall call *higher dimensional uniformisation*. Just as in the case of $n = 2$, this approach will continue to be plagued by accessory parameters. Nevertheless, inspite of this problem, this does provide a better understanding of the meaning of higher dimensional uniformisation. The Teichmüller space related to this uniformisation will be the Teichmüller components of Hitchin, which we shall denote by $\mathcal{T}_n$.

In the usual uniformisation of Riemann surfaces, for the case of genus $g > 1$, the uniformisation theorem tells us that the Riemann surface Σ can be represented by the quotient of the half plane (which is a domain in $\mathbb{CP}^1$) by the action of a Fuchsian subgroup of $PSL(2, \mathbb{R})$. This Fuchsian subgroup is isomorphic to the fundamental group of the Riemann surface. Each point in Teichmüller space corresponds to a different representation of the fundamental group as a Fuchsian group. Generalising this, we expect a surface Ω embedded in CP^{n-1} on which a Fuchsian subgroup Γ of $PSL(n, \mathbb{R})$ which is isomorphic to the the fundamental group of the Riemann surface[4]. One then obtains a W-surface which is given by Ω/Γ. We will show that there is an extension of this surface to a $(n-1)$ dimensional manifold obtained by isomonodromic deformations. This manifold provides us with W-space. This is the primary result of this paper.

The solutions $\{f_i\}$ of (2.3) furnish (polymorphic) maps from the Riemann surface Σ to $\mathbb{CP}^{n-1}$ where f_i are the homogeneous coordinates. Choosing coordinates such that the differential equation has the Laguerre-Forsyth form has a nice interpretation here. This coordinate can be identified with the coordinate of a fundamental domain in the half-plane with Poincaré metric. We shall refer

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[4]Since there does not seem to be a general definition of a Fuchsian subgroup of $PSL(n, \mathbb{R})$, we shall require that it be a subgroup of $PSL(n, \mathbb{R})$ which acts properly discontinuously on Ω with no fixed points. It may be necessary to impose more conditions in the definition of a Fuchsian group but we shall ignore these technicalities for now.

to the map from the half-plane to $\mathbb{CP}^{n-1}$ as the *developing map*.

The developing map furnishes us with a domain $\Omega \subset \mathbb{CP}^{n-1}$ as the image of the half-plane. However, the monodromy homomorphism generically gives a subgroup of $\mathrm{PSL}(n, \mathbb{C})$ which is not exactly what one wants. Nevertheless, the differential equation approach does provide a concrete realisation of how a higher projective structure may be realised on Riemann surfaces (see section 5). Also, for the purposes of our discussion we shall assume that the monodromy homomorphism gives a subgroup Γ of $\mathrm{PSL}(n, \mathbb{R})$ which corresponds to a point in the Teichmüller component $\mathcal{T}_n$. In other words, we shall assume that the differential equation being considered is the one related to the uniformisation problem we are formulating here.

3.1 W-space from isomonodromic deformations

Let $\tilde{f}_i(t_p)$ be extensions of the solutions f_i of the (2.3) such that deformations parametrised by t_p are isomonodromic. As we have shown, this requires that $\tilde{f}$ satisfy equations (2.18). Further, we impose the boundary condition $\tilde{f}(t_i) = f(z)$ at $t_p = 0$ for $i > 1$.

The differential equation gives maps into $\mathbb{CP}^{n-1}$ for every time slice labelled by $\{t_i\}$. The corresponding developing map provides us with $\Omega(t_i)$. The same Fuchsian subgroup Γ of $\mathrm{PSL}(n, \mathbb{R})$ acts on $\Omega(t_i)$ since the deformations are isomonodromic by choice. We can now form the $(n-1)$ dimensional complex manifold[5] $\widetilde{\Omega} \equiv \cup \Omega(t_i) \subset \mathbb{CP}^{n-1}$. Γ continues to act discontinuously on the whole of $\widetilde{\Omega}$. We then obtain a *W-manifold* $W_n(\Sigma)$ of dimension $(n-1)$ by $\widetilde{\Omega}/\Gamma$. Hereafter, we shall refer to the space $\widetilde{\Omega}/\Gamma$ as the *W*-manifold and Ω/Γ as the *W*-surface. It is clear that $\widetilde{\Omega}$ plays the role of the half-plane in the standard uniformisation of Riemann surfaces. As we shall demonstrate later, this manifold provides the setting for the linearisation of *W*-diffeomorphisms. The extended homogeneous coordinates $\tilde{f}_i$ are related to the the Baker-Akhiezer function with only a finite number of non-zero times (see the work of Matsuo for more details[23]).

---
[5]One has to show that $\widetilde{\Omega}$ forms a manifold. As we shall see later, for the case where all u_i vanish, it is easy to see that the $\widetilde{\Omega}$ is a manifold. This will continue to be true for the case $u_i \neq 0$ since diffeomorphisms will not change this.

It is of interest to study the case where all the u_i vanish. The $\tilde{s}_i$ are related to the t_i by the relation $\tilde{s}_i = \frac{z^i}{i!} + t_i + \cdots$ and $\tilde{J} = 1$. The ellipsis refers to terms involving "times" t_j for $j < i$. From this it is clear that the *W*-surface at time t_i has no overlap with the *W*-surface at a different time slice. This implies that $\Omega(t_i) \cap \Omega(t_i') = 0$ for $t_i \neq t_i'$. We expect that this situation will continue to hold for the case $u_i \neq 0$. We shall call the *W*-manifold in this case the *trivial* W-manifold. For example, for the case $n = 3$, the quadratic relation (2.7) gets modified to

$$f_1 f_3 - f_2^2 + 2t f_1^2 = 0 \quad . \tag{3.1}$$

Since $\Omega(t_i)$ are simply connected and are time translations of $\Omega(0)$, $\widetilde{\Omega}$ which is the union of all the $\Omega(t_i)$ is also simply connected. Hence the fundamental group of the *W-manifold* is the same as that of the original Riemann surface. This also suggests that the directions corresponding to t_i for $i > 2$ are probably non-compact.

The *W*-manifold is a hyperbolic manifold (with a non-trivial Kobayashi pseudo-metric induced from the developing map)[24]. It is of interest to see if the pseudo-metric is a metric. This metric however cannot be the Fubini-Study metric. It is possibly a Kahler manifold but we do not know how to establish this. The *W*-manifold described here does not seem to have any relation to the manifold introduced by Zucchini as a possible candidate for *W*-space[25].

Let us define extended inhomogeneous coordinates by $\tilde{s}_i = \tilde{f}_i/\tilde{f}_n$ where $\tilde{f}_i$ are the extensions of f_i. Interpreting $\tilde{s}_i$ as an arbitrary holomorphic coordinate transformation of t_i, eqn. (2.11) has a nice interpretation – it is the $-\frac{1}{n}$-th power of the determinant of the Jacobian of a holomorphic coordinate transformation. This follows from using eqn. (2.18) to replace higher z derivatives of $\tilde{s}_i$ by first order derivatives in t_i and some standard properties of determinants.

$$\tilde{J}^{-\frac{1}{n}} \equiv \tilde{f}_n = \begin{vmatrix} \partial_1 \tilde{s}_1 & \cdots & \partial_{n-1} \tilde{s}_1 \\ \vdots & \ddots & \vdots \\ \partial_1 \tilde{s}_{n-1} & \cdots & \partial_{n-1} \tilde{s}_{n-1} \end{vmatrix}^{-\frac{1}{n}} \quad , \tag{3.2}$$

where $\partial_i = \partial/\partial t_i$ and $\tilde{s}$ are the extended inhomogeneous coordinates. This implies that the homogeneous coordinates transform like tensor densities of weight $-\frac{1}{n}$ on the *W*-manifold.

On the Riemann surface, J transforms as a tensor density of weight $n(n-1)/2$, i.e., it is a section of $K^{\frac{n(n-1)}{2}}$. Similarly, in the higher dimensional case, $\tilde{J}$ can be considered as a section of the determinant bundle of the canonical bundle on the W-manifold. So a field of weight Δ on the Riemann surface which is a section of K^Δ can be lifted to the higher dimensional case as a section of $[2\Delta/(n^2-n)]$-th power of the determinant bundle. This observation should enable us to derive the transformation of arbitrary tensors under W-diffeomorphisms.

In W-space not all diffeomorphisms are allowed. The diffeomorphisms which preserve the form of (2.3) and (2.18) will form the set of restricted diffeomorphisms. Since this has already been recently studied by Gomis *et al.* and we refer the reader to their work[11]. We shall obtain the conditions using a slightly different method in the next section. Gomis *et al.* have also shown that these restricted diffeomorphisms form a semi-group i.e., the composition of two restricted diffeomorphisms gives an restricted diffeomorphism. W-diffeomorphisms are then obtained by projecting the restricted diffeomorphisms to the W-surface.

4 Deformation of complex structure

In this section, we shall first introduce the Beltrami equation in the higher dimensional setting. This encodes the change of complex structure on W-manifolds. The compatibility of this equation with the equations of the KdV hierarchy as well as the differential equation provides restrictions on the allowed Beltrami differentials. In order to relate the W-manifolds to W-gravity, we then show that the higher dimensional Beltrami equation projects to the W-surface as the Beltrami equation which occurs in W-gravity[9, 26]. In order to make this correspondence we discuss an old puzzle raised by Di Francesco *et al.* regarding the relationship of KdV flows and W-diffeomorphism which appears in our setting as the relationship of the Beltrami differentials of higher dimensional uniformisation and the Beltrami differentials of W-gravity. We suggest a resolution to this puzzle and verify it explicitly for the case of W_3 and W_4.

So far we have used the differential equation approach to describe the geometry of W-space. However, for describing Teichmüller space, it better to use other techniques like generalisations of quasi-conformal mappings. Related

to this is the Beltrami equation which describes the deformation of complex structure. In this section, we shall discuss how the Beltrami equation on the W-manifold projects down to the W-surface as a non-linear Beltrami equation. Unlike the case of isomonodromic deformations, a deformation of complex structure leads to a deformation in the Fuchsian group and hence corresponds to a different point in the Teichmüller component $\mathcal{T}_n$. So far we have been implicitly assuming that the $\tilde{f}_i$ are holomorphic functions of t_i i.e.,

$$\bar{\partial}_i \tilde{f} = 0 \quad , \tag{4.1}$$

where $\bar{\partial}_i = \partial/\partial \bar{t}_i$. It is convenient to replace the above holomorphicity condition in terms of the inhomogeneous coordinates since the usual Beltrami equation is written in terms of these coordinates.

$$\bar{\partial}_i \tilde{s} = 0 \quad . \tag{4.2}$$

A deformation of complex structure can be parametrised by means of the Beltrami differentials defined by the following equations

$$[\bar{\partial}_i + \mu_i{}^j \partial_j]\tilde{s} = 0 \quad . \tag{4.3}$$

From this equation, we can show that $\tilde{J}$ satisfies the following equation

$$[\bar{\partial}_i + \mu_i{}^j \partial_j + (\partial_j \mu_i{}^j)]\tilde{J} = 0 \quad . \tag{4.4}$$

We shall only consider a restricted class of Beltrami deformations given by $\mu_i{}^j = 0$ for $i > 1$. This implies that one has $(n-1)$ non-zero components corresponding to $\mu_{\bar{z}}{}^j$. We do not have any geometric insight into why the other $\mu_i{}^j$ vanish but impose it using the standard counting and conformal dimensions of the Beltrami differentials which occur in W-gravity. However it suggests the possibility that the W-manifold is such that it is "rigid" to those complex deformations parametrised by $\mu_i{}^j$ for $i > 1$. Using $\tilde{f}_n = \tilde{J}^{-1/n}$, we obtain the equation

$$[\bar{\partial}_i + \mu_i{}^j \partial_j - \frac{1}{n}(\partial_j \mu_i{}^j)]\tilde{f} = 0 \quad . \tag{4.5}$$

The compatibility of eqn. (4.5) with eqn. (2.18) imposes restrictions on the Beltrami differentials. These are related to the restricted diffeomorphisms obtained

in ref. [11]. These conditions are given by the following set of equations

$$[\partial_p - (L^{p/n})_+ \ , \ \bar{\partial}_i + \mu_i{}^j \partial_j - \frac{1}{n}(\partial_j \mu_i{}^j)]\tilde{f} = 0 \quad . \tag{4.6}$$

For $n = 3$, these conditions are

$$\begin{aligned}
\partial_t \mu &= -\frac{1}{6}\partial_z^3 \rho - 2u_2 \partial_z \rho \quad , \\
\partial_t \rho &= 2\partial_z \mu \quad ,
\end{aligned} \tag{4.7}$$

where $t \equiv t_2$; $\mu_{\bar{z}}{}^z \equiv (\mu - \frac{1}{2}\partial_z \rho)$ and $\mu_{\bar{z}}{}^t \equiv \rho$. These relations are consistent with the conditions on infinitesimal W-diffeomorphisms obtained in ref. [11]. The Beltrami equations become

$$[\partial_{\bar{z}} + (\mu - \frac{1}{2}\partial_z \rho)\partial_z + \rho \partial_t]\tilde{s} = 0 \quad ,$$

$$[\partial_{\bar{z}} + (\mu - \frac{1}{2}\partial_z \rho)\partial_z + \rho \partial_t - \frac{1}{3}\partial_z(\mu - \frac{1}{2}\partial_z \rho) - \frac{1}{3}(\partial_t \rho)]\tilde{f} = 0 \quad . \tag{4.8}$$

Thus equation (4.5) and conditions (4.7) combine to give us the Beltrami equation on the W-manifold.

Projecting to the W-surface, the Beltrami equation becomes a non-linear equation. We obtain

$$[\bar{\partial}_{\bar{z}} + (\mu - \frac{1}{2}\partial_z \rho)\partial_z + \rho(\partial_z^2 - \frac{2}{3J}\partial_z)]\tilde{s} = 0 \quad ,$$

$$[\bar{\partial}_{\bar{z}} + \mu \partial_z - (\partial_z \mu) + \rho(\partial_z^2 + 2u_2) - \frac{1}{2}(\partial_z \rho)\partial_z + \frac{1}{6}(\partial_z^2 \rho)]\tilde{f} = 0 \quad , \tag{4.9}$$

where we have used $\partial_t \tilde{f} = (\partial_z^2 + 2u_2)\tilde{f}$. This is precisely the form of the Beltrami equations obtained in earlier work[9, 26].

We would like to comment on the observation by the authors of [16] that the generators of W-transformations (w_i in our notation) do not in general coincide with the generators of the generalised KdV flows (these correspond to $Res(L^{p/n})$). This is reflected by the fact that the Beltrami differentials corresponding to $\mu_1{}^i$ are not directly the Beltrami differentials in the usual formulation of W-gravity. Nevertheless, we claim that there exists an invertible transformation (possibly non-linear) relating the two sets of Beltrami differentials. This transformation corresponds to converting the $\mu_1{}^i$ to $(-i, 1)$ tensors

(i.e., sections of $K^{-i}\bar{K}$) on projecting to the W-surface. In the example we considered μ and ρ are $(-1, 1)$ and $(-2, 1)$ tensors. This is the dual of the $u_i \to w_i$ transformation. It is easy to see that the transformation of $\mu_1{}^i$ will involve derivatives of all the $\mu_1{}^j$ with $j > i$. It is likely that some of the u_i (atleast u_2) will enter this transformation. This possibility first occurs in the case of W_4 where one could have a term of the type $\mu = \mu_1{}^z + (a\partial_z^2 + bu_2)\mu_1{}^{t_3} + \cdots$ where a and b are some constants. This explains why $Res(L^{3/4}) \neq w_4$. As a check, we set $u_2 = 0$ in the W_4 transformations given in [16] and find that for constant transformations that $X_3 = Y_3$ (using the notation of [16]). It is therefore useful to work out an algorithm to generate this transformation. We hope to address this issue in the future[27].

The Beltrami differentials which parametrise the deformation of complex structure can be constructed using quadratic, cubic and higher order symmetric differentials[5]. Thus, if we identify the space of complex structures as being locally parametrised by these differentials, we immediately recover the result that the dimension of this space is given by $\dim(PSL(n, \mathbb{R}))(2g-2) = (2g-2)(n^2-1)$. This space can be identified with the Teichmuller component of Hitchin labelled $\mathcal{T}_n$. This follows by using the result that the Fuchsian group gets deformed by the Beltrami differentials and the space of Fuchsian groups which uniformise is the Teichmüller component $\mathcal{T}_n$.

5 Projective structures with real holonomy

It has been known for a while now that the generators of W-algebras behave as generalised projective connections. However, the exact sense in which this occurs was not clear. The differential equation constructed using the projective connections did give maps into $\mathbb{CP}^{n-1}$, but the fact that the map is from a one dimensional space to a $(n-1)$ dimensional space did not let one use the standard definition of projective structures except when $n = 2$. However, we have now been able to extend the base space to an $(n-1)$ dimensional space and now the standard definition can be applied.

A projective structure can now be defined as maps from charts on $W_n(\Sigma)$ to $\mathbb{CP}^{n-1}$ with transition functions at the overlap of charts belonging to $PSL(n, \mathbb{R})$. Note that the most general transition function would belong to $PSL(n, \mathbb{C})$ but

we are interested in real projective structures or to be precise, $\mathbb{CP}^{n-1}$ structures with real holonomy. This imposes the reality condition on the transition functions. Given a patch definition of the projective structure, one can easily obtain the corresponding projective connections either by using the Wronskian method (if the projective structure is described in terms of homogeneous coordinates) or the Halphen invariants (if the projective structure is defined in terms of inhomogeneous coordinates).

The converse is a little bit more involved. Given the projective connections u_i, one would like to find the projective coordinates. First, one has to extend the u_i as functions of the KdV times such that they solve the relevant equations of the generalised KdV hierarchy. Then the solutions of the equations (2.3) and (2.18) furnish the projective coordinates f_i. Note that one has implicitly extended to the Riemann surface to the trivial W-manifold.

The Halphen invariants I^r can be also be rewritten in terms of the extended variables. For example, for the case $n = 3$ we obtain

$$
\begin{aligned}
I^2(s_1, s_2; z, t) &= \frac{1}{2}\tilde{J}^{\frac{1}{3}}(\partial_t - \partial_z^2)\tilde{J}^{-\frac{1}{3}} \quad, \\
I^3(s_1, s_2; z, t) &= \frac{1}{2}\tilde{J}^{\frac{1}{3}}(\partial_z^3 - 3\partial_t\partial_z)\tilde{J}^{-\frac{1}{3}} + \frac{3}{2}\partial_z(I^2(s_1, s_2; z, t)) \quad.
\end{aligned}
$$

In this higher dimensional setting, the Halphen invariants now are the right generalisations of the Schwarzian.[6] Unlike the situation in section 2, now the symmetry between the inhomogenous coordinates and the independent variables $(z, t, \ldots)$ has been restored in the I^r. In the form given above, Gomis *et al.* refer to these as W-Schwarzians but we shall continue to call them Halphen invariants. It is actually simpler to use the equations (2.18) to obtain the extended Halphen invariants by substituting $\tilde{J}^{-\frac{1}{n}}$ for f and solving for u_i. The Halphen invariants continue to be invariant under projective transformations even after they are extended by the KdV times. Further, they vanish when the coordinates t_i are chosen such that s_i are related to the t_i as in the trivial W-manifold. In this coordinate system, $\tilde{J} = 1$ and by recursively using the general structure of equations in (2.18), one can see that $u_i = 0$. For the $n = 2$ case,

---

[6]We would like to observe that the occurance of the Schur polynomials (in derivatives) in the expressions for the Halphen invariants. Thus, the Schur polynomials might be useful to generate expressions for the Halphen invariants in the general case.

$u_2 = 0$ corresponded to choosing the uniformising coordinate (on the upper half plane). Similarly, here such a coordinate can be thought of as choosing the uniformising coordinates.

Deformations of projective structure are obtained by deforming the Halphen invariants by symmetric differentials. Hence the space of projective structures can be identified by the space of symmetric differentials just as in the case of $n = 2$, a deformation of a projective structure could be described by a quadratic differential. This can be made more precise by using the correspondence with Higgs bundle description of Teichmüller space[7, 5].

6   Conclusion and outlook

In this paper, we have constructed W-space using the generalised KdV times as w-coordinates. This provides the geometry behind W-gravity and also gives geometrical insight into the Teichmüller components of Hitchin. The new set of complex manifolds $W_n(\Sigma)$ that we have introduced may be interesting in their own right. It would be interesting to obtain more information about them. We believe that they must be Kahler manifolds like the original manifold Σ. What is the metric on this space? It is also of interest to understand the mapping class group (equivalent to understanding "large" W-diffeomorphisms) so that we can construct the moduli spaces for W-gravity. This will be of some significance for W-strings.

Since the moduli of these spaces correspond to variation of Hodge structures[9, 15], there must exist a description of these structures directly in the higher dimension. This can be done via Higgs bundles in higher dimensions which has been studied by Simpson[28]. Assuming this is possible, this would imply that the differential equation (2.3) and the equations (2.18) can be rewritten as self-duality equations in the higher dimensional space. The proposal in [5] of the flat connections as generalised vielbeins and spin-connections can probably be made more concrete. This would lead to a covariant description of W-gravity. We hope to discuss this issue in the future.

One of the important issues is to understand how to couple matter to W-gravity directly in this higher dimensional setting. Since objects in this higher dimensional space project down to terms involving higher derivatives, it is ob-

vious that extrinsic geometry has to play a role. An interesting example has been furnished by the case of a massless rigid particle whose action is given only by the extrinsic curvature term. It has been shown recently that this theory has W_3 symmetry[29]. There must exist stringy generalisations of this with W_n symmetry. This could possibly be related to the QCD string or even new fundamental string theories.

This work has dealt with the case of Riemann surfaces with genus $g > 1$. It is easy to include the cases of the sphere with $n \geq 3$ punctures and the torus with $n > 0$ punctures since they also admit Fuchsian uniformisation. In the differential equation approach studied in this paper, punctures correspond to regular singular points of the differential equation such that the monodromy around the puncture belong to a specific conjugacy class. It is useful to work out this case since one can be a lot more explicit and this can provide us with a better understanding of the w-moduli. Finally, even though we have restricted our discussion to the case of W_n-gravity, the generalisation to include the case of other simple Lie groups is straight forward. This should cover the class of W-gravities which are called "generic". It is not clear to us how the "exotic" algebras fit into this construction.

Appendix

A   Wronskians

We discuss some elementary facts about linear differential equations. Let $\{f_i\}$, $i = 1, \ldots, n$ be a set of linearly independent solutions of a n-th order homogeneous linear differential equation of the form (2.2). Given $\{f_i\}$, one can reconstruct the differential equation (2.2). This is done as follows. Define the *Wronskian* as follows

$$
W \equiv \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ Df_1 & Df_2 & \cdots & Df_n \\ \vdots & \vdots & & \vdots \\ D^{n-1}f_1 & D^{n-1}f_2 & \cdots & D^{n-1}f_n \end{vmatrix} \quad . \tag{A.1}
$$

We shall refer to the Wronskian by $W[f_i]$ or just W when its arguments are obvious. Further, define $W_i \equiv \{W$ with the i-th row replaced by $(D^n f_1, \ D^n f_2, \ \cdots, \ D^n f_n)\}$. Then, $\{f_i\}$ solve equation (2.2) with $\hat{u}_i = -(W_i/W)$. The proof of this is simple. The linear independence of the solutions implies that any other solution f will be a linear combination of the $\{f_i\}$. This can be expressed in the following form from which the differential equation for f is obtained.

$$
\tilde{W} \equiv \begin{vmatrix} f & f_1 & \cdots & f_n \\ Df & Df_1 & \cdots & Df_n \\ \vdots & \vdots & & \vdots \\ D^n f & D^n f_1 & \cdots & D^n f_n \end{vmatrix} = 0 \quad . \tag{A.2}
$$

Comparing equation (A.2) with (2.2), we obtain $\hat{u}_i = -(W_i/W)$.

One can check that under the substitution $f_i = \lambda(z)s_i$, the Wronskian satisfies the following property

$$
W[f_i] = \lambda^n W[s_i] \quad \text{for} \quad i = 1, \ldots, n \quad , \tag{A.3}
$$

for any function $\lambda(z)$.

One can easily see that $W_1 = (-)^{n+1}DW$, which implies that $\tilde{u}_1 = (-)^n D \, ln(W)$. So if $\tilde{u}_1 = 0$ (as in equation (2.3)), then it follows that $DW = 0$ which implies that W is independent of z. From the definition of the Wronskian it follows that $W \in K^{n(j+\frac{n-1}{2})}$ if $f_i \in K^j$. Here K is the canonical holomorphic line bundle on a Riemann surface with coordinate z. For the condition $W =$ constant to make sense globally, we require that W be a scalar ($W \in K^0$) which implies that $f_i \in K^{\frac{1-n}{2}}$.

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